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Translated by M.D.F.

PMM U.S.S.R.,Vol.50,No.4,pp.483-491,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

SPHERICALLY LAYERED INCLUSIONS IN A HOMOGENEOUS ELASTIC MEDIUM*

S.K. KANAUN and L.T. KUDRYAVTSEVA

A three-dimensional homogeneous and isotropic elastic medium is considered that contains an isolated inhomogeneity (inclusion) in the shape of a sphere. It is assumed that the elastic moduli of the medium within the sphere depend only on the distance r to the centre of the inclusion. It is shown that in the case of a constant external field the problem of the equilibrium of a medium with an inhomogeneity reduces to a system of ordinary differential equations in three scalar functions of the variable r. An inhomogeneity with a piecewise-constant dependence of the elastic moduli on r (a spherically layered inclusion) is examined in detail. In this case, an effective calculational algorithm is proposed to construct the solution of the problem. The solution of the problem of one inclusion is then utilized to determine the effective elastic moduli of a medium with a random set of spherically layered inclusions and the estimates of the stress concentration at individual inhomogeneities. The method of an effective (selfconsistent) field is used to take account of interaction between the inclusions.

The problem of a spherically layered inclusion in a homogeneous elastic medium was solved /1-3/ for particular forms of the constant external field. The method proposed below enables us, within the framework of a single scheme, to examine both spherically layered inclusions with practically any number of layers and inclusions with elastic moduli varying continuously along the radius for an arbitrary homogeneous external stress (strain) field.

1. The integral equation of the problem. In an infinite homogeneous medium with the elastic modulus tensor c_0 let there be an isolated inhomogeneity occupying a finite domain V whose characteristic function is $V(x)$, where $x(x_1, x_2, x_3)$ is a point of the medium. We shall consider the elastic modulus tensor $c(x)$ to be a piecewise-smooth function of the coordinates with the domain V. We examine the deformation of the medium $\varepsilon(x)$ under the effect of self-equilibrated forces at infinity and certain mass forces.

Let $\varepsilon_0(x)$ denote the external field of deformations that would exist in a medium when there are no inhomogeneities and the same loading conditions. It is known /4/ that a perturbation of the strain tensor $\varepsilon_1(x) = \varepsilon(x) - \varepsilon_0(x)$ in a medium with an inhomogeneity will satisfy the equation

$$\varepsilon_{1\alpha\beta}(x) + \int_V K_{\alpha\beta\lambda\mu}(x-x')\, c_1^{\lambda\mu\nu\rho}(x')\, \varepsilon_{1\nu\rho}(x')\, dx' = -\int_V K_{\alpha\beta\lambda\mu}(x-x')\, c_1^{\lambda\mu\nu\rho}(x')\, \varepsilon_{0\nu\rho}(x')\, dx', \quad c_1(x) \rightleftharpoons c(x) - c_0 \tag{1.1}$$

The kernel $K(x)$ of the integral operator K in this equation is expressed in terms of the second derivatives of Green's function $G(x)$ for the medium c_0

$$K_{\alpha\beta\lambda\mu}(x) = -(\nabla_\alpha \nabla_\lambda G_{\beta\mu}(x))_{(\alpha\beta)(\lambda\mu)} \tag{1.2}$$

The function $G(x)$ satisfies the well-known equation (δ_β^α is the Kronecker delta, and $\delta(x)$ is the delta function)

$$\nabla_\alpha c_0^{\alpha\beta\lambda\mu} \nabla_\lambda G_{\mu\nu}(x) = -\delta_\nu^\beta \delta(x) \tag{1.3}$$

*Prikl.Matem.Mekhan.,50,4,633-643,1986

The Fourier transformation $K^*(k)$ of the function $K(x)$ (the symbol of the operator $\mathbf{K}$) is a homogeneous function of zero degree in k and by virtue of (1.2), (1.3) has the form

$$K^*_{\alpha\beta\lambda\mu}(k) = (k_\alpha k_\lambda G^*_{\beta\mu}(k))_{(\alpha\beta)(\lambda\mu)} \tag{1.4}$$
$$G^*(k) = L^{-1}(k), \quad L^{\alpha\beta}(k) = k_\lambda c_0^{\lambda\alpha\beta\mu} k_\mu$$

As follows from (1.1), the field $\varepsilon_1(x)$ outside the inclusion is restored uniquely by means of its values within the domain V. The equation for the function $\varepsilon_1^+(x) = \varepsilon_1(x) V(x)$ is obtained by multiplying both sides of (1.1) by $V(x)$. The solution of such equations for a non-degenerate, bounded elastic modulus tensor of the inclusion $c(x)$ exists and is unique for a fairly broad class of right sides (see /5/ for details). If the surfaces of discontinuity of the piecewise-smooth function $c_1(x)$ are closed, do not intersect, and do not contain angular points and edges, then the solution of (1.1) will be a bounded piecewise-smooth function that decreases as $|x|^{-3}$ at infinity.

Consider an elastic isotropic medium. In this case it is convenient to introduce a tensor basis whose elements are collected from the divalent tensor $\delta_{\alpha\beta}$ and the unit vector n_α /6/ in order to represent the quadrivalent tensors in the problem:

$$E_{1\alpha\beta\lambda\mu} = {}^1/_2(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda}), \quad E_{2\alpha\beta\lambda\mu} = \delta_{\alpha\beta}\delta_{\lambda\mu} \tag{1.5}$$
$$E_{3\alpha\beta\lambda\mu} = \delta_{\alpha\beta} n_\lambda n_\mu, \quad E_{4\alpha\beta\lambda\mu} = n_\alpha n_\beta \delta_{\lambda\mu}$$
$$E_{5\alpha\beta\lambda\mu} = {}^1/_4(\delta_{\alpha\lambda} n_\beta n_\mu + \delta_{\beta\lambda} n_\alpha n_\mu + \delta_{\alpha\mu} n_\beta n_\lambda + \delta_{\beta\mu} n_\alpha n_\lambda)$$
$$E_{6\alpha\beta\lambda\mu} = n_\alpha n_\beta n_\lambda n_\mu$$

These six linearly independent tensors, which are symmetric in the first and second pair of subscripts, form a closed algebra with respect to multiplication, convolutions over two subscripts. We will later denote this operation by a dot

$$(E_i \cdot E_j)_{\alpha\beta\lambda\mu} = E_{i\alpha\beta}^{\ \ \ \nu\rho} \cdot\cdot E_{j\nu\rho\lambda\mu}$$

(see the multiplication table of the tensors E_i in /6/, Appendix 4).

The representation of the tensors c_0 and c_1 in the basis (1.5) has the form

$$c_0 = \lambda_0 E_2 + 2\mu_0 E_1, \; c_1(x) = \lambda_1(x) E_2 + 2\mu_1(x) E_1$$

where λ_0, μ_0 are the Lamé coefficients of the medium, $\lambda(x) = \lambda_0 + \lambda_1(x)$, and $\mu(x) = \mu_0 + \mu_1(x)$ are the same quantities for the inclusion. The tensor $K^*(k)$ in (1.4) is determined in the case of an isotropic medium by the relationship

$$K^*(n) = \frac{1}{\mu_0}(E_5(n) - \varkappa_0 E_6(n)), \quad \varkappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}, \quad n = \frac{k}{|k|} \tag{1.6}$$

2. Special representation of the operator K. We consider a special representation of the singular integral operator $\mathbf{K}$ in (1.1), which will be important later. We introduce the spherical coordinate system (r, n), where $r = |x|$, $n = x/|x|$ is a vector on the unit sphere Ω_1. Let $f^*(s, n)$ denote the Mellin transform of the tensor function $f(r, n)$ in the variable r. The following formulas hold /7/

$$f^*(s, n) = \int_0^\infty r^{s-1} f(r, n)\, dr, \quad f(r, n) = \frac{1}{2\pi i}\int_{\tau - i\infty}^{\tau + i\infty} r^{-s} f^*(s, n)\, ds \tag{2.1}$$

As follows from the results in /8/, the operator $\mathbf{K}$ allows of the following representation in finite, piecewise-smooth functions $f(r, n)$:

$$(\mathbf{K}\cdot f)(r, n) = \frac{1}{2\pi i}\int_{\tau - i\infty}^{\tau + i\infty} r^{-s} (\mathbf{K}_s \cdot f^*)(s, n)\, ds, \quad \tau = \frac{1}{2} \tag{2.2}$$

where the operator $\mathbf{K}_s$ is defined by the relationship

$$(\mathbf{K}_s \cdot f^*)(s, n) = -\frac{i}{(2\pi)^3}\Gamma(3 - s)\Gamma(s)\int_{\Omega_1} d\Omega_m \int_{\Omega_1} (-n\cdot m)^{-s}(l\cdot m)^{s-3} K^*(m) \cdot f^*(s, l)\, d\Omega_l \tag{2.3}$$

Here $\Gamma(s)$ is the Euler Gamma function, n, m, l are vectors on the unit sphere, and $K^*(m)$ has the form (1.6).

We will find the result of the action of the operator $\mathbf{K}_s$ on elements of the tensor basis (1.5). To evaluate the integrals on the right-hand side of (2.3) we use formulas that have been obtained by direct integration after having passed to angular coordinates on the unit sphere $(\mathrm{Re}\, s < 1)$

$$\int_{\Omega_1} (n\cdot m)^{-s}\, d\Omega_m = I, \quad \int_{\Omega_1} (n\cdot m)^{-s}\, m_\alpha m_\beta d\Omega_m = \frac{I}{3-s}(\delta_{\alpha\beta} - n_\alpha n_\beta)$$

$$\int_{\Omega_1} (n\cdot m)^{-s} E_6(m)\, d\Omega_m = \frac{I}{(3-s)(5-s)}[E_2 + 2E_1 - s(E_3(n) + E_4(n) + 4E_5(n)) + s(2+s)E_6(n)]; \quad I = 2\pi\frac{1+e^{-i\pi s}}{1-s}$$

From here and (2.3) it follows that

$$\mathbf{K}_s\cdot E_1 = \frac{1}{\mu_0(3-s)(5-s)}[T_1 + (1-\varkappa_0)T_3], \tag{2.4}$$

$$\mathbf{K}_s\cdot E_2 = \frac{1-\varkappa_0}{\mu_0}\frac{1}{3-s}T_2$$

$$\mathbf{K}_s\cdot E_3 = \frac{1-\varkappa_0}{\mu_0}\left(\frac{1}{3-s}T_2 - \frac{1}{5-s}T_3\right),$$

$$\mathbf{K}_s\cdot E_4 = -\frac{1-\varkappa_0}{\mu_0}\frac{2-s}{s(3-s)}T_2$$

$$\mathbf{K}_s\cdot E_5 = -\frac{1}{2\mu_0(3-s)(5-s)}\{(1-s)[T_1 + (1-\varkappa_0)T_3] + (3-s)(1-\varkappa_0)T_3\}$$

$$\mathbf{K}_s\cdot E_6 = \frac{s-2}{\mu_0 s(2+s)}\left\{(1-\varkappa_0)\left(\frac{1}{3-s}T_2 - \frac{3-s}{5-s}T_3\right) + \frac{2}{(3-s)(5-s)}[T_1 + (1-\varkappa_0)T_3]\right\}$$

Therefore, all six tensors of $\mathbf{K}_s\cdot E_i$ are expressed in terms of three linearly-independent tensors T_j that have the form

$$T_1 = (E_1 - sE_5)(5-s) - T_3,\quad T_2 = E_2 - sE_4 \tag{2.5}$$
$$T_3 = E_2 + 2E_1 - s(E_3 + E_4 + 4E_5) + s(s+2)E_6$$

We note that T_j are eigenelements of the operator $\mathbf{K}_s$

$$\mathbf{K}_s\cdot T_1 = \frac{1}{2\mu_0}T_1,\quad \mathbf{K}_s\cdot T_2 = \frac{1-\varkappa_0}{\mu_0}T_2,\quad \mathbf{K}_s\cdot T_3 = \frac{1-\varkappa_0}{\mu_0}T_3$$

The last equations follow from (2.4) and (2.5).

3. A spherically symmetric inhomogeneity. Let the Lame coefficients λ and μ of the inclusion be functions of the variable r only. We will examine the solution of (1.1) in the case when the medium is loaded by a constant external field ε_0. By virtue of the linearity of the problem, the tensor $\varepsilon_1(r,n)$ is represented in the form

$$\varepsilon_{1\alpha\beta}(r,n) = A_{\alpha\beta}^{\lambda\mu}(r,n)\,\varepsilon_{0\lambda\mu} \tag{3.1}$$

where the tensor $A_{\alpha\beta}^{\lambda\mu}$ can be considered to be symmetric in the upper and lower pairs of indices. The equation for $A(r,n)$ follows from (1.1) and has the following form without super or subscripts.

$$A(r,n) + (\mathbf{K}\cdot c_1\cdot A)(r,n) = -(K\cdot c_1)(r,n) \tag{3.2}$$

We shall seek its solution as a linear combination of the tensors $E_i(n)$ (1.5) with scalar coefficients dependent only on r. Then the product $c_1\cdot A$ can be represented in the form

$$(c_1\cdot A)(r,n) = \sum_{i=1}^{6} S_i(r)E_i(n) \tag{3.3}$$

where $S_i(r)$ are scalar functions of r. We substitute this expression for $c_1\cdot A$ into (3.2) and apply a Mellin transform to both sides of the relationship obtained. Taking (2.2) into account, we obtain

$$A^*(s,n) + \sum_{i=1}^{6} S_i^*(s)(\mathbf{K}_s\cdot E_i)(s,n) = -(\mathbf{K}_s\cdot c_1^*)(s,n) \tag{3.4}$$
$$c_1^*(s) = \lambda_1^*(s)E_2 + 2\mu_1^*(s)E_1$$

It follows from (2.4) that the tensors $\mathbf{K}_s\cdot E_i$ and $\mathbf{K}_s\cdot c_1^*$ are linear combinations of three tensors T_j (2.5). But then the tensor $A^*(s,n)$ should naturally also be sought in the form of the same linear combination

$$A^*(s,n) = \sum_{j=1}^{3} \alpha_j^*(s)T_j(s,n) \tag{3.5}$$

Here $\alpha_j^*(s)$ are scalar functions of the Mellin transform parameter s, whose r - representation is $\alpha_j(r)$.

Since the operation $D = rd/dr$ in the initial r-space corresponds to multiplication by $-s$ in the Mellin transform space, it follows from (2.5) and (3.5) that the expression for the tensor $A(r, n)$ has the form

$$A(r, n) = (E_1 + E_5(n) D)(5 + D) d_1(r) + (E_2 + E_4(n) D) \alpha_2(r) + [E_2 + 2E_1 + (E_3(n) + E_4(n) + 4E_5(n)) D + E_6(n) D(D - 2)] (\alpha_3(r) - \alpha_1(r)) \tag{3.6}$$

If the functions $\alpha_j(r)$ can be selected such that $A(r, n)$ satisfies (3.2), then by virtue of the uniqueness, the tensor (3.6) is a solution of the problem under consideration.

We will now construct the function $\alpha_j(r)$. We substitute the tensor $A^*(s, n)$ (3.5) into (3.4) and take account of (2.4). Equating coefficients on the right and left sides of the relationship obtained for the linearly independent tensors T_1 and T_3, we arrive at the following formulas relating the functions α_1 and α_3 after some reduction

$$\begin{gathered} s(s+2)(s-3)(s-5)\alpha_1^*(s) + \frac{1}{2\mu_0}\Phi_1^*(s) = -2s(s+2)\frac{\mu_1^*(s)}{\mu_0} \\ s(s+2)(s-3)(s-5)[\alpha_3^*(s) - (1-\varkappa_0)\alpha_1^*(s)] + \frac{1-\varkappa_0}{\mu_0}\Phi_3^*(s) = 0 \\ \Phi_1^*(s) = 2s(s+2)S_1^*(s) + (s+2)(s-1)S_5^*(s) + 4(s-2)S_6^*(s) \\ \Phi_3^*(s) = (s-3)[2(s+2)S_3^*(s) + (s+2)S_5^*(s) + 2(s-2)S_6^*(s)] \end{gathered} \tag{3.7}$$

Here $S_i^*(s)$ is the Mellin transform of the scalar coefficients $S_i(r)$ in expansion (3.3). Substituting (3.6) into (3.3), we obtain the following expressions for the functions $S_i(r)$ in (3.7)

$$\begin{gathered} S_1 = 2\mu_1[(3+D)\alpha_1 + 2\alpha_3], \quad S_3 = \lambda_1(5+D)D\alpha_3 + 2\mu_1 D(\alpha_3 - \alpha_1) \\ S_5 = 2\mu_1 D[(1+D)\alpha_1 + 4\alpha_3], \quad S_6 = 2\mu_1 D(D-2)(\alpha_3 - \alpha_1) \end{gathered} \tag{3.8}$$

The equality of the coefficients for T_2 will yield a relationship in which the function α_2 occurs. However, it will later be more convenient to consider the function

$$\beta(r) = \alpha_2(r) + (5 + D)\alpha_3(r)$$

instead of α_2.

We obtain a relationship analogous to (3.7) for β by multiplying both sides of (3.4) on the right by the tensor E_2 and by taking account of the equalities

$$\begin{gathered} A \cdot E_2 = (E_2 + E_4 D)\beta, \quad c_1 \cdot A \cdot E_2 = S_7 E_2 + S_8 E_1 \\ S_7 = \lambda_1(3 + D)\beta + 2\mu_1\beta, \quad S_8 = 2\mu_1 D\beta \end{gathered} \tag{3.9}$$

Using (2.4), we have in the same way as above

$$\begin{gathered} s(s-3)\beta^*(s) - \frac{1-\varkappa_0}{\mu_0}\Phi_2^*(s) = \frac{1-\varkappa_0}{\mu_0} s(3\lambda_1^*(s) + 2\mu_1^*(s)) \\ \Phi_2^*(s) = sS_7^*(s) + (s-2)S_8^*(s) \end{gathered} \tag{3.10}$$

Differential equations which the desired functions will satisfy are not difficult to obtain from the preceding relationships. To do this, we should go over to the r-representations in (3.7) and (3.10) by replacing the Mellin transforms of $\alpha_1, \alpha_3, \beta$ and S_i by their originals, and the parameter $(-s)$ by the differential operator D. For instance, let $\lambda_1(r)$ and $\mu_1(r)$ be finite functions with piecewise-continuous second derivatives and $d\lambda_1/dr = d\mu_1/dr = 0$ for $r = 0$. Then from (3.7) and (3.8) we obtain a system of two ordinary fourth-order differential equations for the functions α_1 and α_3 and from (3.10) a second-order equation for β whose right sides and coefficients are piecewise-continuous functions. The solution of these equations should be bounded everywhere and should satisfy the conditions

$$\begin{gathered} D\alpha_i = D^2\alpha_i = 0, \; i = 1, 3; \; D\beta = 0 \text{ for } r = 0 \\ \alpha_1, \alpha_3, \beta \to 0 \text{ as } r \to \infty \end{gathered} \tag{3.11}$$

The first group of these conditions is satisfied of the continuity of the function $A(r, n)$ for $r = 0$ and the second group because $A(r, n)$ tends to zero at infinity.

4. A spherically layered inhomogeneity. Let $\lambda_1(r)$ and $\mu_1(r)$ be finite piecewise-constant functions with discontinuities at the points $r = a_i, i = 1, 2, \ldots, N$; $0 < a_1 < a_2 < \ldots < a_N$. In this case the inclusion consists of a kernel and the $(N-1)$-th spherical layer within which the elastic moduli are constant. Going over from the relationships (3.7) and (3.10) to differential equations for the functions α_1, α_3 and β we obtain that these equations take the following form in the domains of constant λ_1 and μ_1

$$D(D-2)(D+3)(D+5)\alpha_j=0,\ j=1,\ 3;\ D(D+3)\beta=0 \tag{4.1}$$

Writing the general solution of these equations, we obtain that the form of the functions α_1, α_3 and β is determined in the intervals $a_{i-1}<r<a_i$, $i=1, 2, \ldots, N+1$; $a_0=0$, $a_{N+1}=\infty$, by the relationships

$$\begin{aligned}\alpha_1(r)&=Y_1^{(i)}+Y_2^{(i)}r^2+Y_3^{(i)}r^{-3}+Y_4^{(i)}r^{-5}\\ \alpha_3(r)&=Y_5^{(i)}+Y_6^{(i)}r^2+Y_7^{(i)}r^{-3}+Y_8^{(i)}r^{-5},\quad \beta(r)=Y_9^{(i)}+Y_{10}^{(i)}r^{-3}\end{aligned} \tag{4.2}$$

where $Y_j^{(i)}$ are arbitrary constants. Therefore, within each layer the solution of the problem is determined apart from ten constants.

We will now investigate the discontinuities of the derivatives of the functions α_1, α_3 and β on the boundaries of the layers. The absence of singular components of the tensors $\varepsilon(r, n)$ and $A(r, n)$ follows from the continuity of the elastic displacement vector in the whole space. From this and from (3.6) for $A(r, n)$ it is seen that the function β should be continuous while α_1 and α_3 are continuous together with the first derivatives.

Furthermore, let $\mu_1(r)$ be a piecewise-constant function with discontinuities at the points $r=a_i$ $(i=1, 2, \ldots, N)$ that equal zero for $r>a_N$. Its Mellin transform $\mu_1^*(s)$ (2.1) has the form

$$\mu_1^*(s)=-\frac{1}{s}\sum_{i=1}^{N}[\mu]_i a_i^s \tag{4.3}$$

where the quantity $[\varphi]_i$ for any piecewise-continuous function $\varphi(r)$ is determined by the relationship

$$[\varphi]_i=\varphi(a_i+0)-\varphi(a_i-0),\quad \varphi(a_i\pm 0)=\lim_{\varepsilon\to 0}\varphi(a_i\pm\varepsilon),\quad \varepsilon>0$$

By using integration by parts, the equalities

$$\begin{aligned}(\mu_1 D\beta)^*(s)&=-\sum_{i=1}^{N}[\mu_1\beta]_i a_i^s-s(\mu_1\beta)^*(s)\\ (\mu_1 D^2\alpha_j)^*(s)&=-\sum_{i=1}^{N}([\mu_1 D\alpha_j]_i-s[\mu_1\alpha_j]_i a_i^s+s^2(\mu_1\alpha_j)^*(s)\end{aligned} \tag{4.4}$$

can be obtained from (2.1) for the continuous function β and the functions α_j that are continuous together with the first derivatives.

Taking account of (4.3) and (4.4), the function $\Phi_1^*(s)$ in (3.7) is representable in the form

$$\begin{aligned}\Phi_1^*(s)=2\sum_{i=1}^{N}\{&(s+2)(s^2-8s+9)[\mu_1\alpha_1]_i-4(s+2)[\mu_1\alpha_3]_i-\\ &(s^2-3s+6)[\mu_1 D\alpha_1]_i-4(s-2)[\mu_1 D\alpha_3]_i\}a_i^s+s(s+2)(s-3)(s-5)(\mu_1\alpha_1)^*(s)\end{aligned} \tag{4.5}$$

The equation

$$\begin{aligned}s(s+2)(s-3)(s-5)(\mu\alpha_1)^*(s)=-\sum_{i=1}^{N}\{&s^3[\mu\alpha_1]_i-s[\mu(6+D)\alpha_1]_i+\\ &s[\mu(D^2+6D-1)\alpha_1]_i-[\mu(D^3+6D^2-D-30)\alpha_1]_i\}a_i^s\\ \mu=\mu_0+\mu_1\end{aligned} \tag{4.6}$$

holds for a function $\alpha_1(r)$ of the form (4.2).

Substituting (4.5) into (3.8) and taking account of (4.6), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\{&s^3\mu_0[\alpha_1]_i-s^2\mu_0[(6+D)\alpha_1]_i+s([\mu(D^2+6D-1)\alpha_1]_i-\\ &[\mu_1(7\alpha_1-3D\alpha_1+4\alpha_3+4D\alpha_3)]_i)-[\mu(D^3+6D^2-D-\\ &30)\alpha_1]_i+2[\mu_1(4\alpha_3-4D\alpha_3-9\alpha_1+3D\alpha_1)]_i\}a_i^s=-2\sum_{i=1}^{N}(2+s)[\mu]_i a_i^s\end{aligned}$$

Equating factors for the linearly independent functions a_i^s on the left and right sides of this relationship, we arrive at N equalities, each of which connects two polynomials. The coefficients of identical powers of s in these polynomials should be equal. Hence, after algebraic reduction we obtain the following system of conditions on the jumps of the function $\alpha_1(r)$ and its derivatives at the points $r=a_i$ on the layer boundaries

$$\begin{aligned}&[\alpha_1]_i=0,\ [D\alpha_1]_i=0\\ &[\mu D^2\alpha_1]_i=-2[\mu]_i-3[\mu(2+D)\alpha_1]_i-4[\mu(1+D)\alpha_3]_i\\ &[\mu D^3\alpha_1]_i=16[\mu]_i+[\mu(48+25D)\alpha_1]_i+16[\mu(2+D)\alpha_3]_i\end{aligned} \tag{4.7}$$

Analogous conditions can be obtained for the jump of the functions α_3 and β and their derivatives by the same means from (3.10) and the second relationship in (3.7)

$$[\alpha_3]_i = 0,\ [D\alpha_3]_i = 0 \tag{4.8}$$
$$[(\lambda + 2\mu) D^2\alpha_3]_i = -2[\mu]_i - 6[\mu(1 + D)\alpha_1]_i - 4[\mu\alpha_3]_i - [(5\lambda + 6\mu) D\alpha_3]_i$$
$$[(\lambda + 2\mu) D^3\alpha_3]_i = 16[\mu]_i + 24[\mu(2 + D)\alpha_1]_i + 32[\mu\alpha_3]_i + [(25\lambda + 42\mu) D\alpha_3]_i$$
$$[\beta]_i = 0,\ [(\lambda + 2\mu) D\beta]_i = -[3\lambda + 2\mu]_i - [(3\lambda + 2\mu)\beta]_i$$

We note that part of the conditions (4.7) and (4.8) can be obtained by utilizing a general expression for the jump in the strain tensor on the boundary of two elastic media /9/

$$[\varepsilon]_i = -K^*_{i+1}(n)\cdot[c]_i\cdot\varepsilon(a_i - 0) \tag{4.9}$$

The tensor $K^*(n)$ is defined by relation (1.6) in which the parameters λ_0 and μ_0 should be replaced by λ_{i+1}, μ_{i+1}. Substituting $\varepsilon = (E_1 + A)\cdot\varepsilon_0$ here, where A has the form (3.6), expressions can be obtained for the jumps in the functions α_1, α_3, their first two derivatives and the function β that agree with (4.7) and (4.8). Eqs.(4.7) and (4.8) for the jumps in the third derivatives of α_1, α_3 and the first derivative of β do not follow from (4.9) and are specific for the problem under consideration.

All the constants $Y_j^{(i)}$ and in (4.2) for the functions α_1, α_3 and β can be found from the relationships (4.7), (4.8) and their boundary conditions at zero and infinity. In the special case of a homogeneous inclusion $(N = 1)$, the equalities $Y_i^{(1)} = 0\ (i = 3, 4, 7, 8, 10)$, $Y_i^{(2)} = 0\ (i = 1, 2, 5, 6, 9)$ follow from (3.11). The remaining ten constants in (4.2) are determined from conditions (4.7) and (4.8) and have the form

$$Y_1^{(1)} = \frac{2[\mu]_1(\lambda_0 + 2\mu_0)}{(15\mu_0 - 6[\mu]_1)(\lambda_0 + 2\mu_0) - 4\mu_0[\mu]_1},\quad Y_5^{(1)} = \frac{\mu_0}{\lambda_0 + 2\mu_0} Y_1^{(1)}$$
$$Y_9^{(1)} = \frac{3[\lambda + \mu]_1}{3(\lambda_0 + 2\mu_0) - [3\lambda + 2\mu]_i},\quad Y_2^{(1)} = Y_6^{(1)} = 0$$
$$Y_3^{(2)} = \frac{5}{2}a^3Y_1^{(1)},\quad Y_4^{(2)} = -\frac{3}{2}a^5Y_1^{(1)},\quad Y_7^{(2)} = \frac{5}{2}a^3Y_5^{(1)}$$
$$Y_8^{(2)} = -\frac{3}{2}a^5Y_5^{(1)},\quad Y_{10}^{(2)} = a^3Y_9^{(1)}$$

Substituting these values of $Y_j^{(i)}$ into (4.2) and the result into (3.6), we arrive at the well-known solution /10/.

We will now construct the algorithm to calculate the constants $Y_j^{(i)}$ in the general case.

5. An algorithm for the numerical solution of the problem. We introduce the $N + 1$ ten-dimensional vectors $Y^{(i)}$ whose components are the constants $Y_j^{(i)}$ governing the solution (4.2) in the i-th interval (in the i-th layer) and the $N + 1$ vectors $X^{(i)}(r)$ with the components $(a_{i-1} < r < a_i)$

$$X_1^{(i)}(r) = \alpha_1(r),\quad X_2^{(i)}(r) = D\alpha_1(r),\quad X_3^{(i)}(r) = D^2\alpha_1(r) \tag{5.1}$$
$$X_4^{(i)}(r) = D^3\alpha_1 r$$
$$X_5^{(i)}(r) = \alpha_3(r),\quad X_6^{(i)}(r) = D\alpha_3(r),\quad X_7^{(i)}(r) = D^2\alpha_3(r)$$
$$X_8^{(i)}(r) = D^3\alpha_3(r)$$
$$X_9^{(i)}(r) = \beta(r),\quad X_{10}^{(i)}(r) = D\beta(r)$$

It follows from (4.2) that the vectors $Y^{(i)}$ and $X^{(i)}(r)$ are connected by the relationships

$$X^{(i)}(r) = H(r)Y^{(i)},\quad Y^{(i)} = H^{-1}(r)X^{(i)}(r); \tag{5.2}$$
$$H = h_1 \oplus h_1 \oplus h_2$$
$$h_1 = \begin{Vmatrix} 1 & r^2 & r^{-3} & r^{-5} \\ 0 & 2r^2 & -3r^{-3} & -5r^{-5} \\ 0 & 4r^2 & 9r^{-3} & 25r^{-5} \\ 0 & 8r^2 & -27r^{-3} & -125r^{-5} \end{Vmatrix},\quad h_2 = \begin{Vmatrix} 1 & r^{-3} \\ 0 & -3r^{-3} \end{Vmatrix}$$

Therefore, the vector $Y^{(i)}$ and, therefore, the solution (4.2) within the i-th interval, is uniquely defined by the value of the vector $X^{(i)}(r)$ at any point $r\ (a_{i-1} < r < a_i)$. If the value of the vector $X^{(i)}(r)$ is known at the point $r = a_{i-1} + 0$ (at the left end of the i-th interval), its value at the right end, for $r = a_i - 0$ is determined by virtue of (5.2) by the formula

$$X^{(i)}(a_i) = R^{(i)}X^{(i)}(a_{i-1}),\quad R^{(i)} = H(a_i)H^{-1}(a_{i-1}) \tag{5.3}$$

where we call $R^{(i)}$ the transfer matrix.

It follows from relationships (4.7) and (4.8) that the vectors $X^{(i)}$ and $X^{(i+1)}$ at the point a_i on the boundary of the i-th and $(i+1)$-th intervals are connected by the relationship

$$X^{(i+1)}(a_i) = F^{(i)} + \Gamma^{(i)} X^{(i)}(a_i) \tag{5.4}$$

It is easy to restore the form of the transition matrix $\Gamma^{(i)}$ and the transition vector $F^{(i)}$ from (4.7) and (4.8).

Suppose the vector of the solution in the first interval $X^{(1)}(a_1)$ is known. Then the vectors $X^{(i+1)}(a_i)$ governing the solution in the $(i+1)$-th interval, is expressed, by virtue of (5.3) and (5.4), in terms of the vector $X^{(1)}(a_1)$

$$X^{(i+1)}(a_i) = g^{(i)} + G^{(i)} X^{(1)}(a_1) \tag{5.5}$$

$$g^{(1)} = F^{(1)}, \quad g^{(i)} = F^{(i)} + \sum_{j=1}^{i-1} \Big(\prod_{k=i}^{j+1} Q^{(k)} \Big) F^{(j)}, \quad i = 2, 3, \ldots, N$$

$$G^{(i)} = \prod_{k=i}^{1} Q^{(k)}, \quad Q^{(k)} = \Gamma^{(k)} R^{(k)}$$

Here $R^{(1)}$ is the unit matrix, and the transfer matrices $R^{(k)}$ $(k = 2, 3, \ldots, N)$ are defined in (5.3).

We will utilize the boundary conditions to construct the vector $X^{(1)}(a_1)$. It follows from the boundedness of the solution at $r = 0$ that expressions (4.2) for the functions α_1, α_3 and β do not contain negative powers of r in the first interval, i.e., $Y_k^{(1)} = 0$ $(k = 3, 4, 7, 8, 10)$. Then a relation exists between the components of the vector $X^{(1)}$ which can be represented in the form

$$X^{(1)} = M Z^{(1)}, \quad Z^{(i)} = P_1 X^{(i)} \tag{5.6}$$

Here $Z^{(i)}$ are column-vectors with components $X_j^{(i)}$ $(j = 1, 2, 5, 6, 9)$ and the matrices M (10×5) and P_1 (5×10) have the form

$$M = \left\| \begin{array}{c|c|c} m_1 & 0 & 0 \\ \hline m_2 & 0 & 0 \\ \hline 0 & m_1 & 0 \\ \hline 0 & m_2 & 0 \\ \hline 0 & 0 & m_3 \end{array} \right\|, \quad P_1 = \left\| \begin{array}{c|c|c|c|c} m_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & m_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & m_4 \end{array} \right\|$$

$$m_1 = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad m_2 = \left\| \begin{matrix} 0 & 2 \\ 0 & 4 \end{matrix} \right\|, \quad m_3 = \left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|, \quad m_4 = \| 1, 0 \|$$

It follows from the fact that the functions α_1, α_3 and β approach zero at infinity that their expressions (4.2) in the $(N+1)$-th interval (in the medium) contain only negative powers of r, i.e., $Y_j^{(N+1)} = 0$ $(j = 1, 2, 5, 6, 9)$. It can hence be shown that a relationship exists between the components of the vector $X^{(N+1)}$ which can be represented in the form

$$W^{(N+1)} = L Z^{(N+1)}, \quad W^{(i)} = P_2 X^{(i)} \tag{5.7}$$

Here $W^{(i)}$ is a vector with the components $X_k^{(i)}$ $(k = 3, 4, 7, 8, 10)$ and the matrices L (5×5) and P_2 (5×10) have the form

$$L = l_1 \oplus l_1 \oplus l_2, \quad P_2 = \left\| \begin{array}{c|c|c|c|c} 0 & m_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & m_1 & 0 \\ \hline 0 & 0 & 0 & 0 & m_5 \end{array} \right\|$$

$$l_1 = \left\| \begin{matrix} -15 & -8 \\ 120 & 49 \end{matrix} \right\|, \quad l_2 = -3, \quad m_5 = \| 0 \ \ 1 \|$$

In order to find the components of the vector $Z^{(1)}$ we substitute the expression for $X^{(N+1)}(a_N)$ from (5.5) into (5.7) and take account of (5.6). We consequently arrive at the linear equation

$$BZ^{(1)} = f; \; B = (P_2 - LP_1) G^{(N)} M, \; f = (LP_1 - P_2)\, g^{(N)} \tag{5.8}$$

(the matrix $G^{(N)}$ and the vector $g^{(N)}$ are defined by the relationship (5.5)). Solving this equation, we find the vectors $X^{(i+1)}(a_i)$ $(i = 1, 2, \ldots, N)$ from relationships (5.5) and (5.6), and then the vectors of the constants $Y^{(i)}$ which determine the solution within all the layers, from (5.2).

The algorithm described was the basis for a computer solution of the problem. An an illustration of the calculation, we examine an inclusion of unit radius consisting of N

spherical layers of thickness $1/N$. Young's modulus $E^{(i)}$ within the i-th layer is given by the formula

$$E^{(i)} = E(a_i),\ i = 1, 2, \ldots, N+1$$
$$E(r) = E_0\left[1 + \delta \exp\left(\frac{\lambda r^2}{r^2-1}\right)\right] \text{ for } r<1,\quad E(r) = E_0 \text{ for } r>1 \tag{5.9}$$

where a_i is the external radius of the i-th layer. Poisson's ratio for the medium and all the layers was taken to be 0.4. The external loading was uniaxial tension along the axis x_3. The stress distribution σ_{33} along an axis orthogonal to x_3 with origin at the centre of the inclusion is shown in Figs.1 and 2 for $N = 60$. The computation was performed for $\delta = -1$; $\lambda = 0, 10^{-1}, 1, 10$ (compliant inclusions, Fig.1), $\delta = 100$ and the same values of λ (rigid inclusions, Fig.2).

As $N \to \infty$, Young's modulus within the inclusion evidently tends to the continuous distribution (5.9). To verify the possibilities of the algorithm the computation was carried out for increasing values of N until a stable stress distribution pattern was obtained. It turned out that for $N > 40$ the distribution is practically invariant as N increases.

6. Effective elastic moduli of a medium with spherically layered inclusions.

In a homogeneous elastic medium (matrix) let a random set of identical spherically-layered inclusions be distributed spatially, uniformly, and isotropically. We consider the problem of calculating the effective elastic moduli of such a composite material and of estimating the mean stress concentration at the individual inclusions. We use the effective (self-consistent) field method /11, 12/. The main assumption of the method is that each inclusion in the composite material will behave as though isolated in a homogeneous medium (matrix) subjected to an effective external strain field ε_* (the stress is $\sigma_* = c_0 \cdot \varepsilon_*$). The field ε_* is made up of the external field ε_0 and the field induced by the surrounding inhomogeneities. This assumption enables a selfconsistent equation to be obtained to determine ε_*, whose solution has the form /12/

$$\varepsilon_* = \Lambda \cdot \varepsilon_0,\quad \Lambda^{-1} = E_1 - n_0 (c_0 \cdot K_0 - E_1) \cdot P$$
$$P = c_0^{-1} \cdot \int_V c_1(x) \cdot (E_1 + A(x))\, dx,\quad K_0 = \frac{1}{4\pi} \int_{\Omega_1} K^*(n)\, d\Omega_n$$

Here n_0 is the numerical concentration of the inclusions; the integral P is evaluated over the volume of the inclusions; and the tensors $A(x)$ and $K^*(n)$ have the form (3.6) and (1.6), respectively.

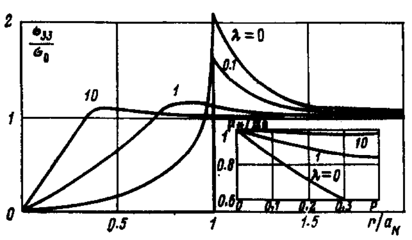

Fig.1

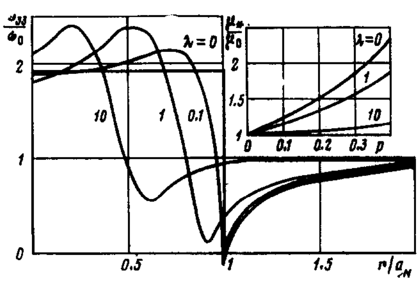

Fig.2

In the case of isotropic media and inclusions, the expression for Λ takes the form

$$\Lambda = d_1 E_2 + d_2 \left(E_1 - \frac{1}{3} E_2\right)$$
$$d_1 = \frac{1}{3}\left(1 + n_0 q_1 \frac{12\mu_0}{3k_0 + 4\mu_0}\right)^{-1},\quad d_2 = \left(1 + n_0 q_2 \frac{9k_0 + 8\mu_0}{5(3k_0 + 4\mu_0)}\right)^{-1}$$

Here k_0 is the matrix bulk modulus, and the coefficients q_1, q_2 are connected with the constants $Y_j^{(i)}$ in (4.2) by the relationships

$$q_1 = \frac{4\pi}{9} \sum_{i=1}^{N} \frac{k_1^{i}}{k_0} (1 + Y_9^{(i)})(a_i^3 - a_{i-1}^3),\quad k_1^{(i)} = k^{(i)} - k_0 \tag{6.1}$$

$$q_2 = \frac{4\pi}{3} \sum_{i=1}^{N} \frac{\mu_1^{(i)}}{\mu_0} (1 + 3Y_1^{(i)} + 2Y_5^{(i)})(a_i^3 - a_{i-1}^3) + \frac{7}{5} \frac{\mu_1^{(i)}}{\mu_0} (3Y_2^{(i)} + 2Y_6^{(i)})(a_i^5 - a_{i-1}^5)$$

($k^{(i)}$ is the bulk modulus of the i-th layer).

Within the framework of the effective-field method, the deformations in the neighbourhood of an arbitrary inclusion are calculated by means of the formula $\varepsilon(x) = (E_1 + A(x))\cdot\varepsilon_*$, and the stresses by Hooke's law.

It follows from the results /12/ that the tensor of the composite's effective elastic moduli c_* in the case under consideration has the form (the coefficients q_1, q_2 are defined by the relationship (6.1))

$$c_* = c_0\cdot(E_1 - n_0 P\cdot\Lambda)^{-1} = k_* E_2 + 2\mu_*(E_1 - {}^1/_3 E_2)$$
$$\frac{k_*}{k_0} = 1 + 3n_0 q_1\left(1 - n_0 q_1 \frac{9k_0}{3k_0 + 4\mu_0}\right)^{-1}$$
$$\frac{\mu_*}{\mu_0} = 1 + n_0 q_2\left(1 - n_0 q_2 \frac{6(k_0 + 2\mu_0)}{5(3k_0 + 2\mu_0)}\right)^{-1}$$

Curves of the effective shear modulus μ_* of the composite material as a function of the volume concentration $p = {}^4/_3\pi n_0 a_N^3$ of the inclusions, whose properties are described at the end of Sect.5 are presented on the right in Figs.1 and 2.

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Translated by M.D.F.